155GISE

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MATHEMATICAL CARTOGRAPHY

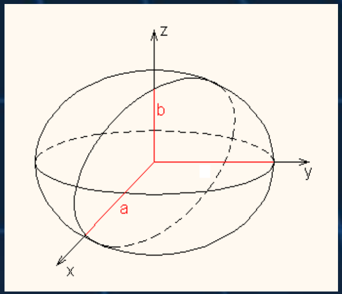
Mathematical cartography is a part of cartography, dealing with mathematical and geometrical basics of cartographic works.

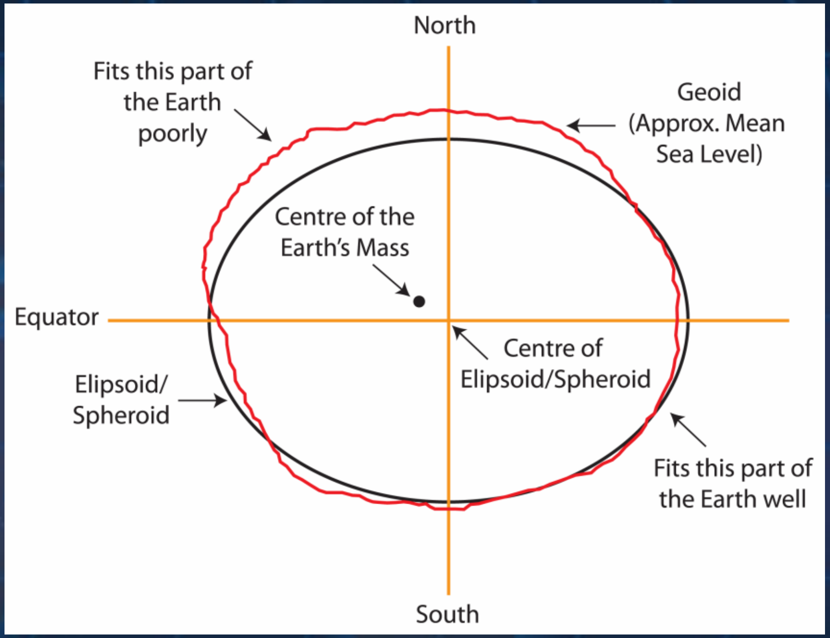
The main goal of mathematical cartography – creation of a continuous planar image of the Earth

Reference surfaces

- the real Earth is too complex in shape
- We need to replace the earth with a mathematically simply defined surface:
  - reference ellipsoid
  - reference sphere
  - plane

- **Reference ellipsoid**
  - **semimajor and semiminor axes (*a*, *b*)**
  - **flattening *i* = *(a–b) / a* (the Earth has 1 / *i* ~ 300)**
  - **Bessel (1841)**
  - **Krasovsky (1940)**
  - **WGS84 (1984)**





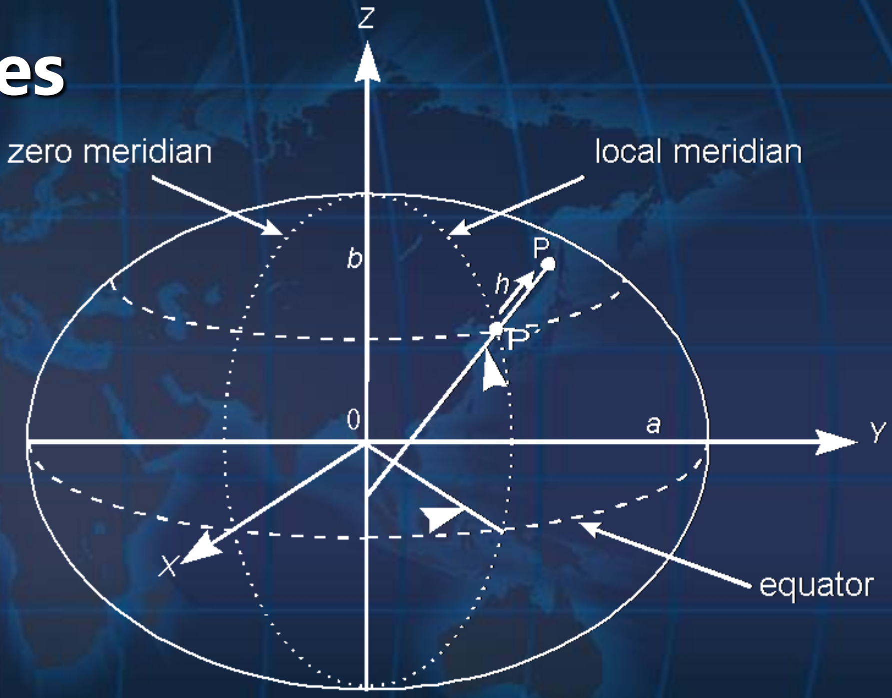
	a	b	i
Bessel	6,377,397 m	6,356,079 m	1 : 299.15
Krasovsky	6,378,245 m	6,356,863 m	1 : 298.30
WGS84	6,378,137 m	6,356,752 m	1 : 298.26

- **Reference sphere**
  - radius (R)
  - for local (up to 300 km) or global ellipsoid replacement
  - dual projections (ellipsoid → sphere → plane)
- **Plane**
  - geographically roughly an area up to 20×20 km
  - altitude differences cannot be neglected

Coordinate systems

- **Geographic coordinates**

- **Ellipsoid (*φ*, *λ*)**
- **Sphere (*U*, *V*)**

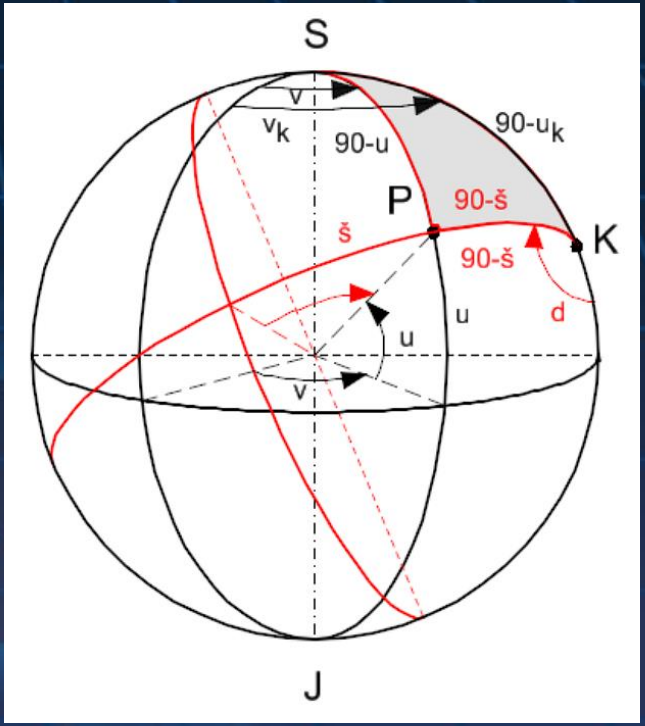


- **The latitude of a point P is the angle between the normal to the reference surface at the point P and the plane of the equator.**
- **The longitude of point P is the angle formed by the plane determined by the earth's axis and point P with a similar plane passing through the base point. (Ferro, Greenwich)**

- # Cartographic coordinates

- Sphere (*s*, *d*) – instead of U and V

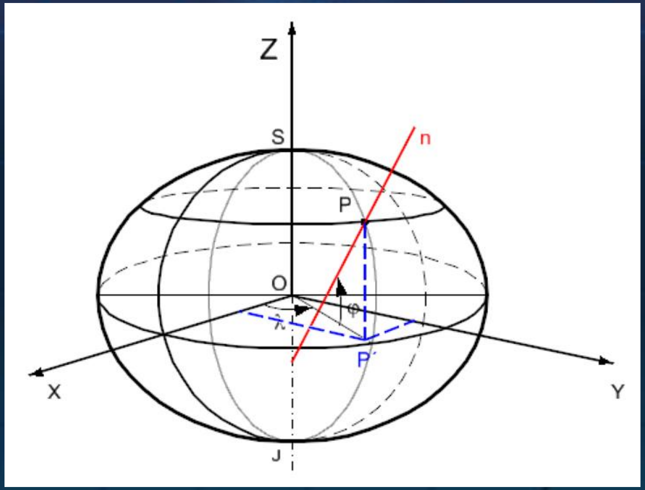
  These are transformed geographic coordinates using princples of the spherical trigonometry



- # 3D rectangular coordinates

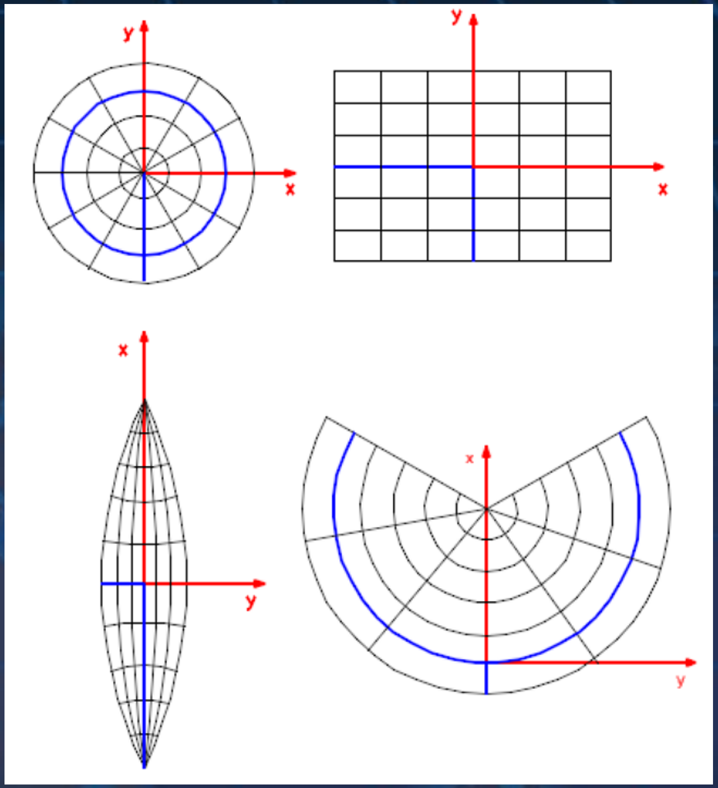
- Origin at the center of the ellipsoid

  X, Y, Z axes
  Z in the axis of rotation of the ellipsoid
  X passes through the point where the plane of the equator intersects the prime meridian
  Y is perpendicular to X, Z



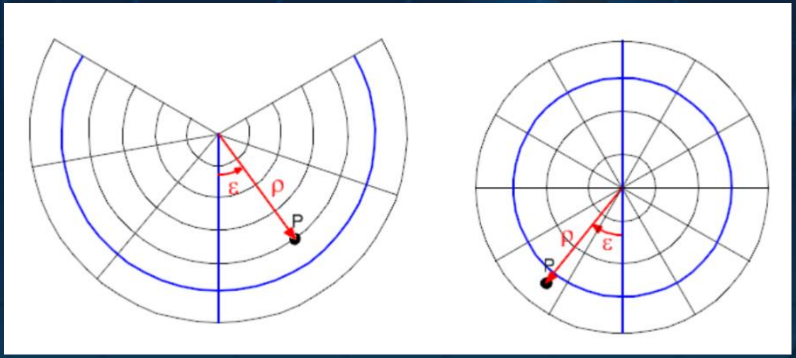
- **Plane coordinates**
  - **rectangular**

    (x, y)



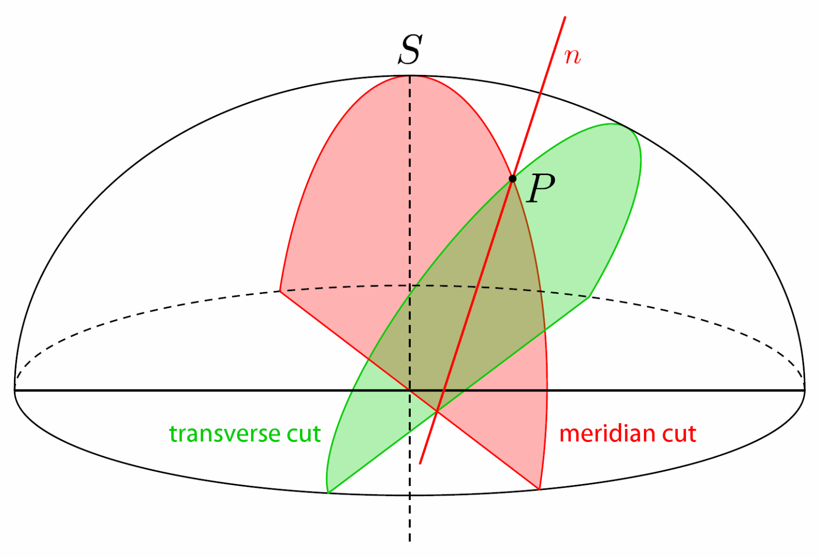
  - **polar**

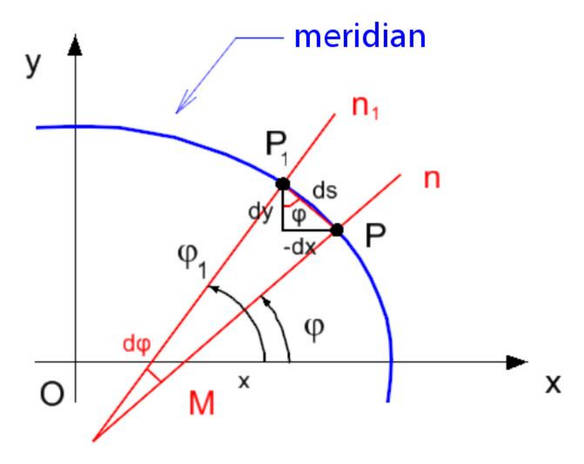
    (ρ, ε)

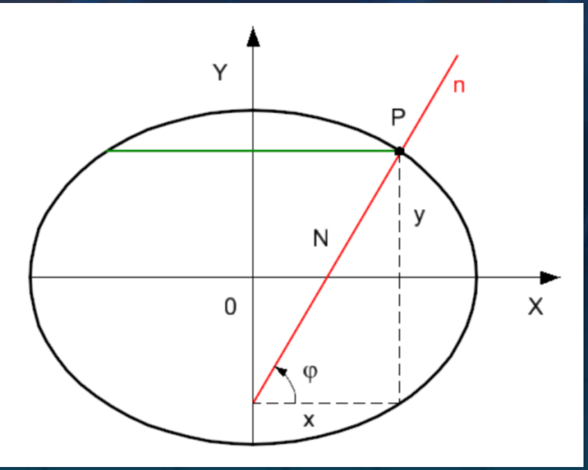


Curvature cuts on an ellipsoid

- **Meridian cut**
  – radius of curvature M
- **Transverse cut**
  – radius of curvature N







Important curves

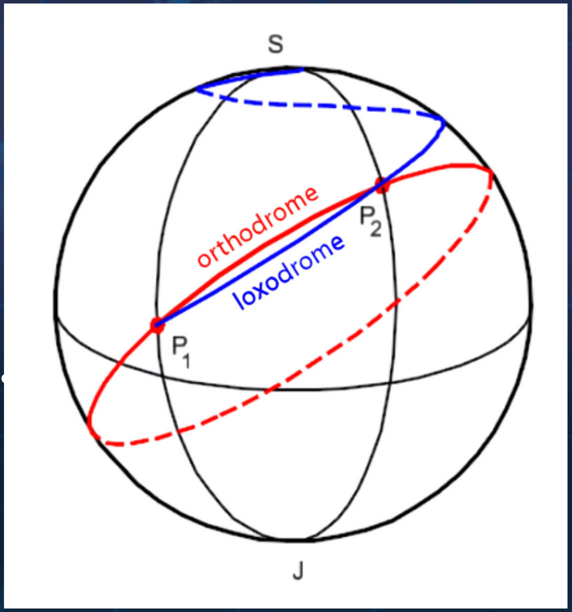
- There are important curves that follow the surface of the reference plane.
- They are used in navigation, maritime or air transport.
- In selected cartographic projections, they are shown as lines/segments, these projections were used in the past for maritime navigation.

- A geodesic curve (ellipsoid), on a sphere called a great circle (orthodrome)
- A loxodrome

Loxodrome

- A curve that intersects meridians under constant azimuth A, the length is infinite.
- It is not the shortest line connecting two points on the reference surface, it is displayed as a general curve in cartographic projections.

Orthodrome (geodetic curve)



- A closed (on a sphere) curve, representing the shortest connecting line of two points along a ref. surface.
- It is part of the great circle.
- It has infinite length on the ellipsoid.
- Clairaut's theorem applies: $\cos\varphi \sin A = \cos\varphi_{MAX}$

Cartographic projection

- Cartographic projection represents the mutual assignment of the position of two points on different reference surfaces.

(In selected cases this can be done geometrically.)

- The projection is uniquely given by its equations

$X = f(\varphi, \lambda)$

$Y = g(\varphi, \lambda)$

Cartographic distortions

- **different reference surfaces have different curvature**
- **distortions occur during projection**

- **lengthwise (length ratio) m_A**
- **area (area ratio) m_P**
- **angular (angle difference) m_ω**

Classification of cartographic projections

1. according to projection properties (distortion)
2. according to the projection area and its position

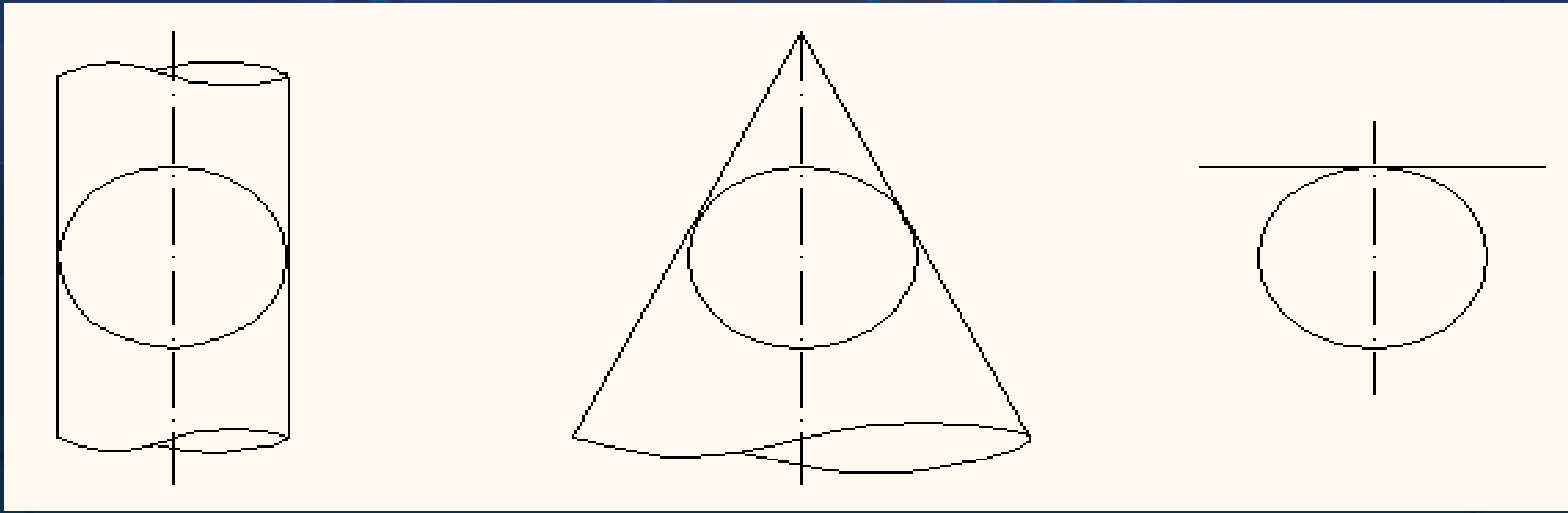
1. Projections divided by distortion

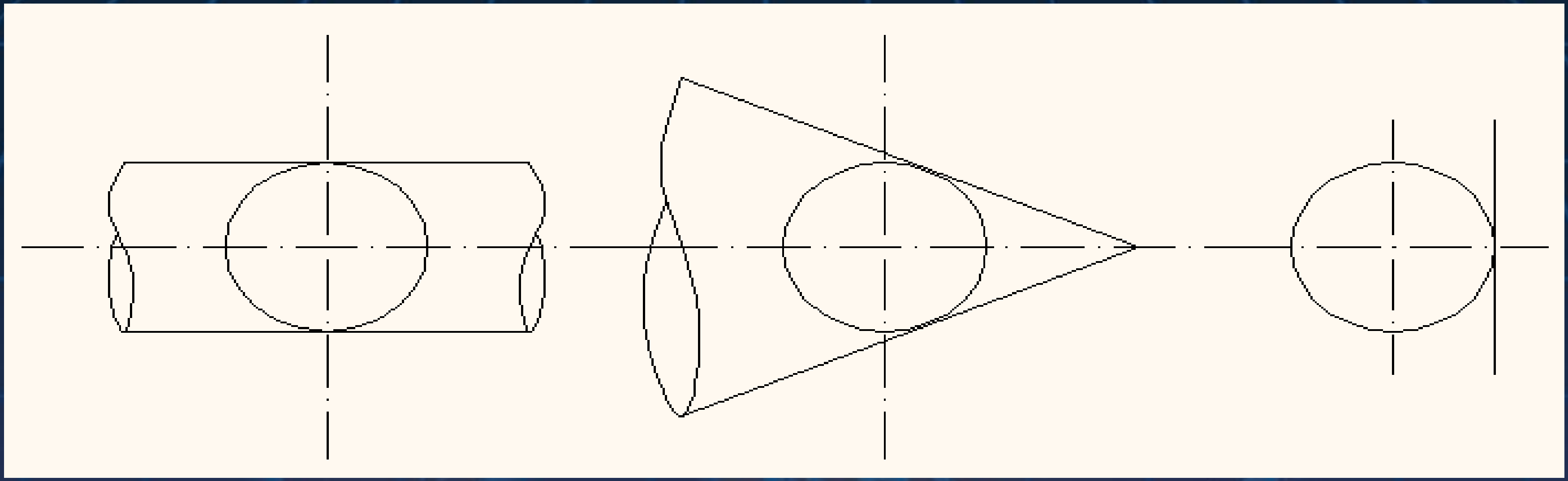
- equidistant (does not distort lengths in certain directions)
- equivalent (does not distort surfaces)
- conformal (does not distort angles)

2. Projections divided by projection area

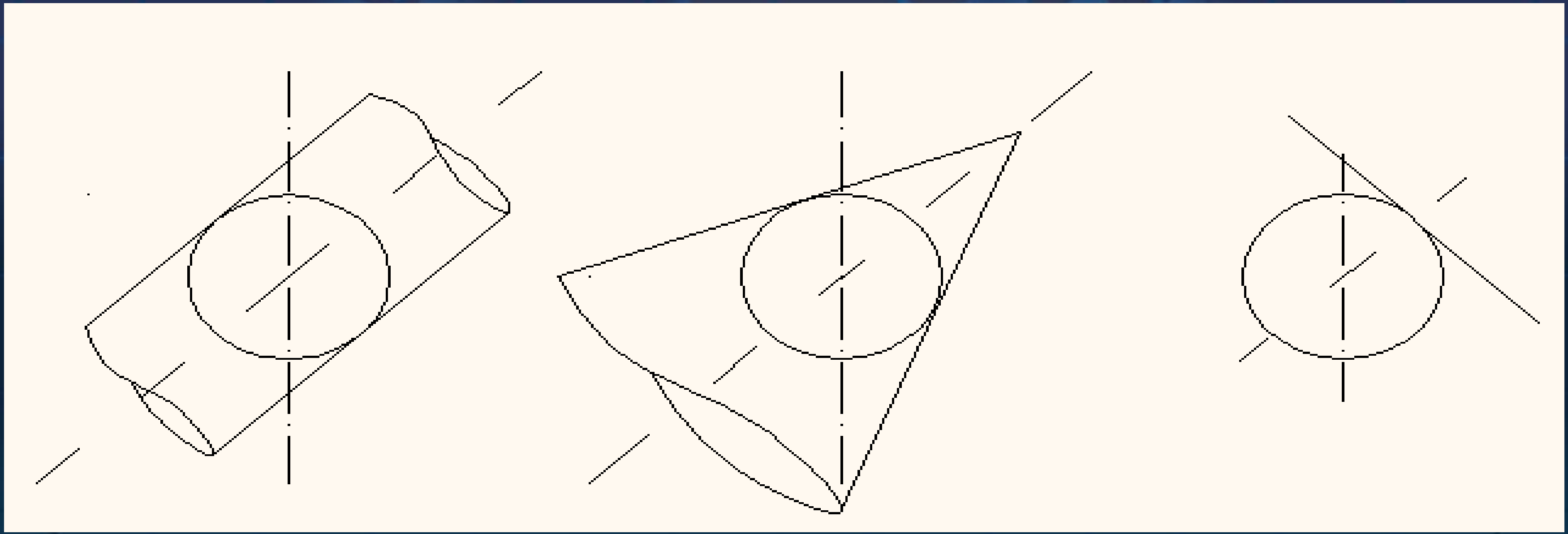
- projections of an ellipsoid on a sphere
- simple projections (projecting on expandable surfaces)
  - conic, cylindrical, azimuthal
- pseudo-projections
  - conic, cylindrical, azimuthal
- polyconic
- polyhedral
- unclassified

- Simple projections
  according to the position of the projection area
  - in normal position
  - in a transverse position
  - in an oblique position





Transverse position



Oblique position

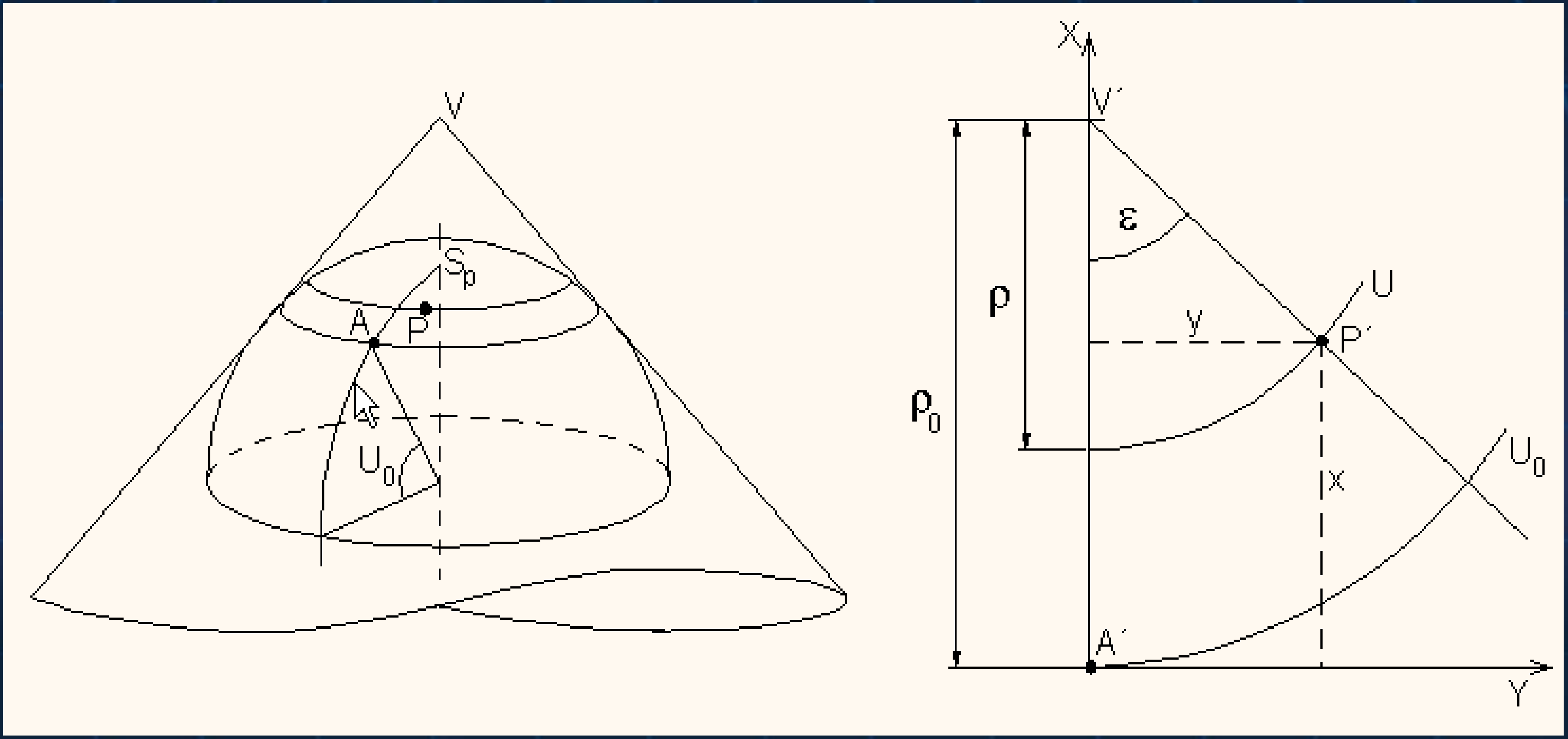
Projection of an ellipsoid on a sphere

- for small map scales we can replace the ellipsoid with a sphere
- we can choose several conditions when deriving the projection equations
  - preserved geographic coordinates
  - projection on a concentric sphere
  - conformity
    - Undistorted prime meridian condition
    - Preserved geographic grid
  - equidistant projection
    - In meridians
    - In parallels
  - equivalent projection

Simple projections

- Plane coordinates can be expressed using a function of only one coordinate
  - e.g. for normal position
    - X = $n\,V$
    - Y = f(U)
- A simple rendition of cartographic meridians and parallels
  - meridians (bundle or grid of straight lines)
  - parallels (straight lines or circles)

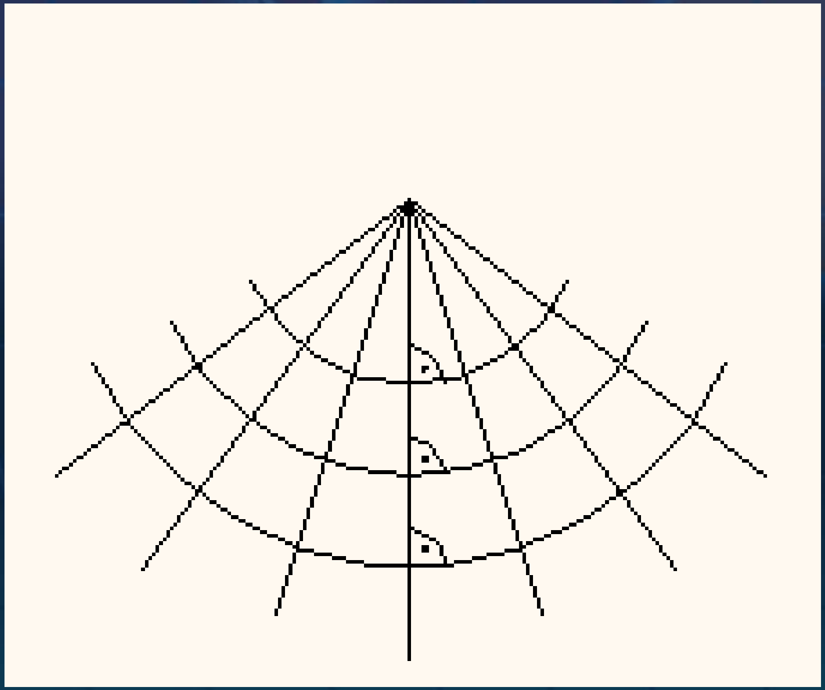
Conic projections



- base parallel – approximately in the center of the area
- prime meridian – from which longitude is calculated

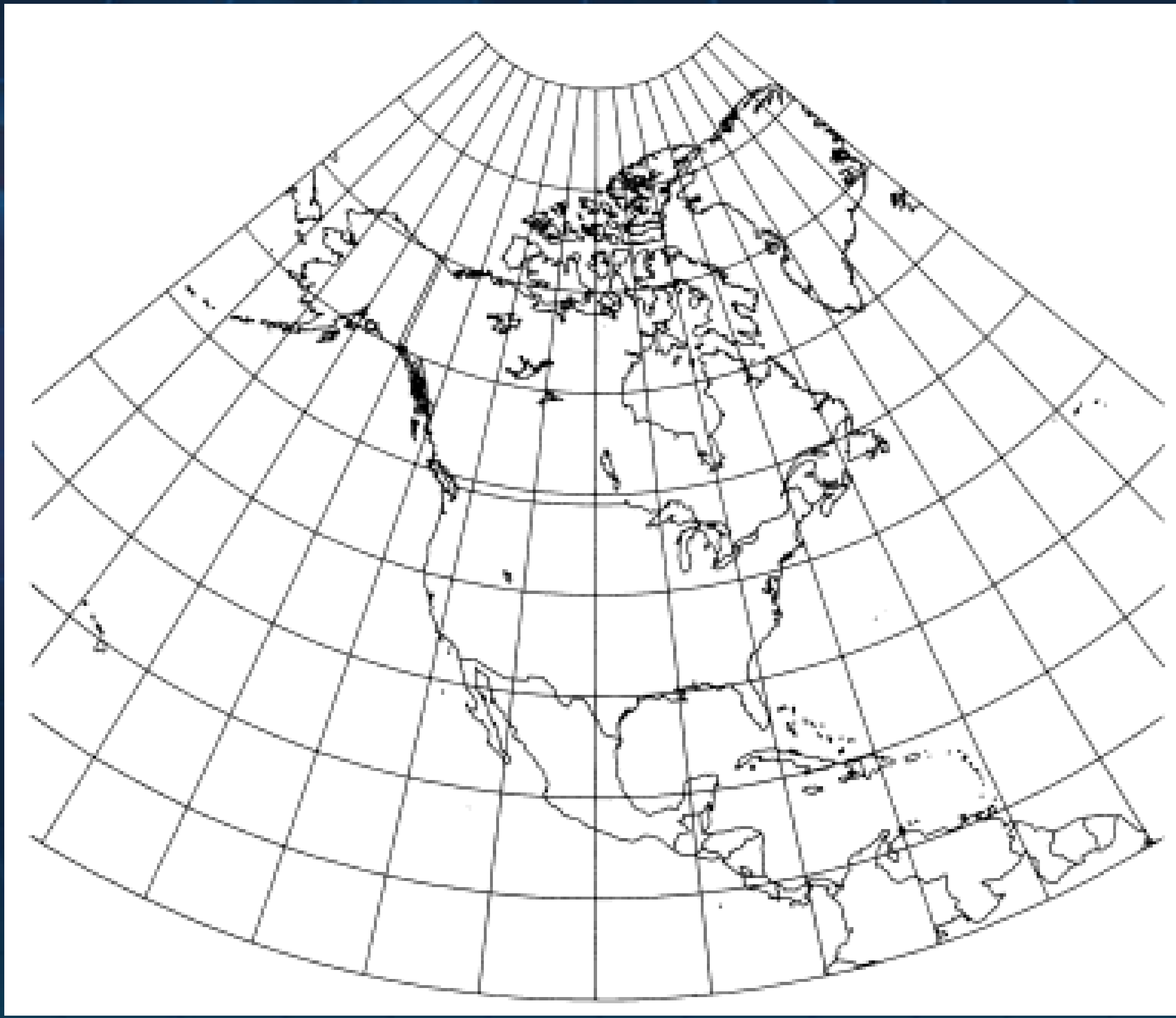
Geographic grid shape

- meridians – a bundle of straight lines
- parallels – concentric circles
- pole – a circle or point

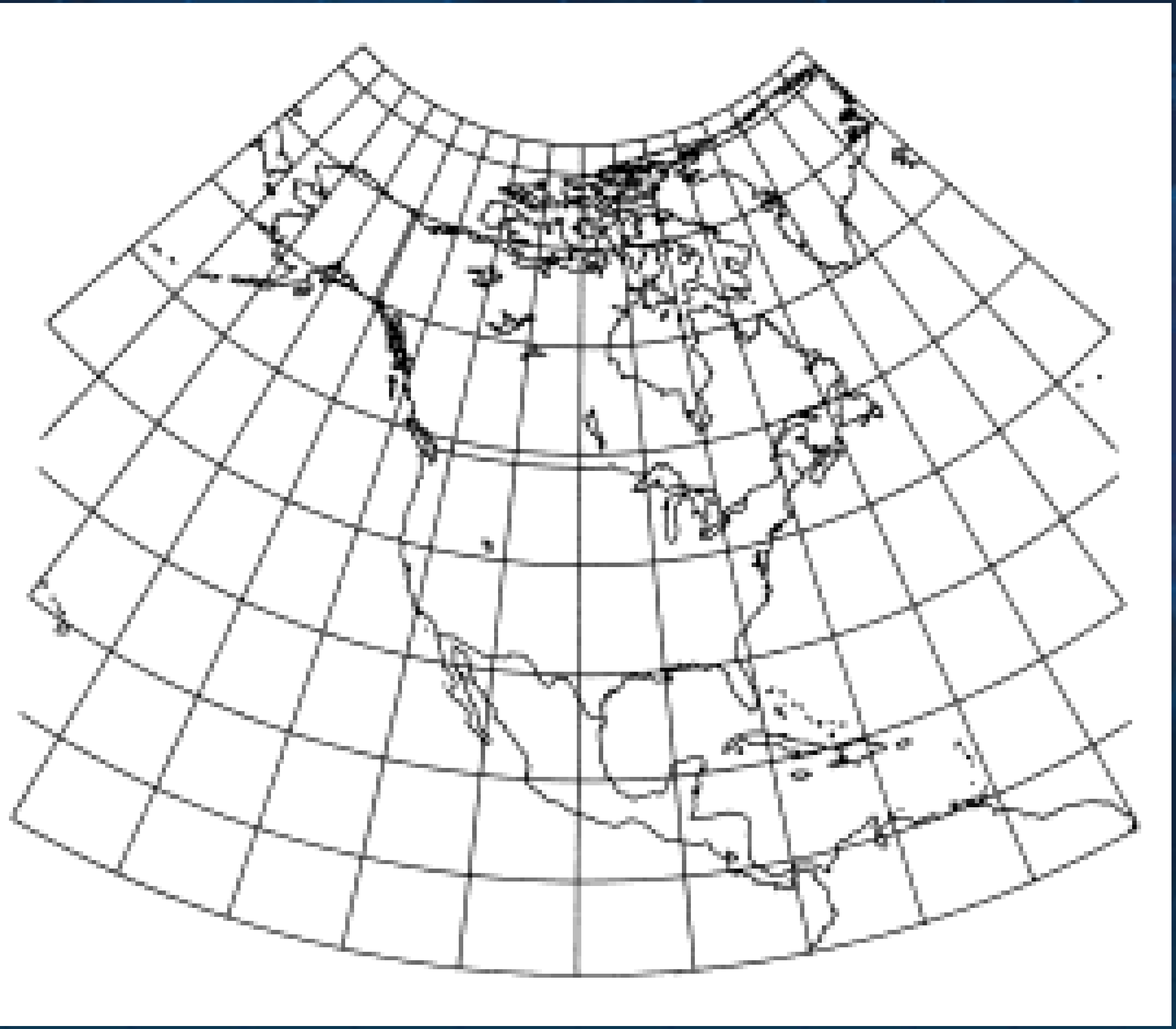


- suitable for territories that are distributed along the small circles on the globe, e.g. spherical belts with a smaller width
- they are mainly used for maps of smaller scales, especially in the normal position (transverse position is unsuitable, in these cases cylindrical views are preferred)
- on the other hand, these projections are completely unsuitable for maps of the entire world in a continuous presentation (opposite polar rendition, distortion changes)
- using a suitable choice of constants we get individual conic representations
  (equidistant, equivalent, conformal ×
  number of undistorted parallels, rendition of the pole)

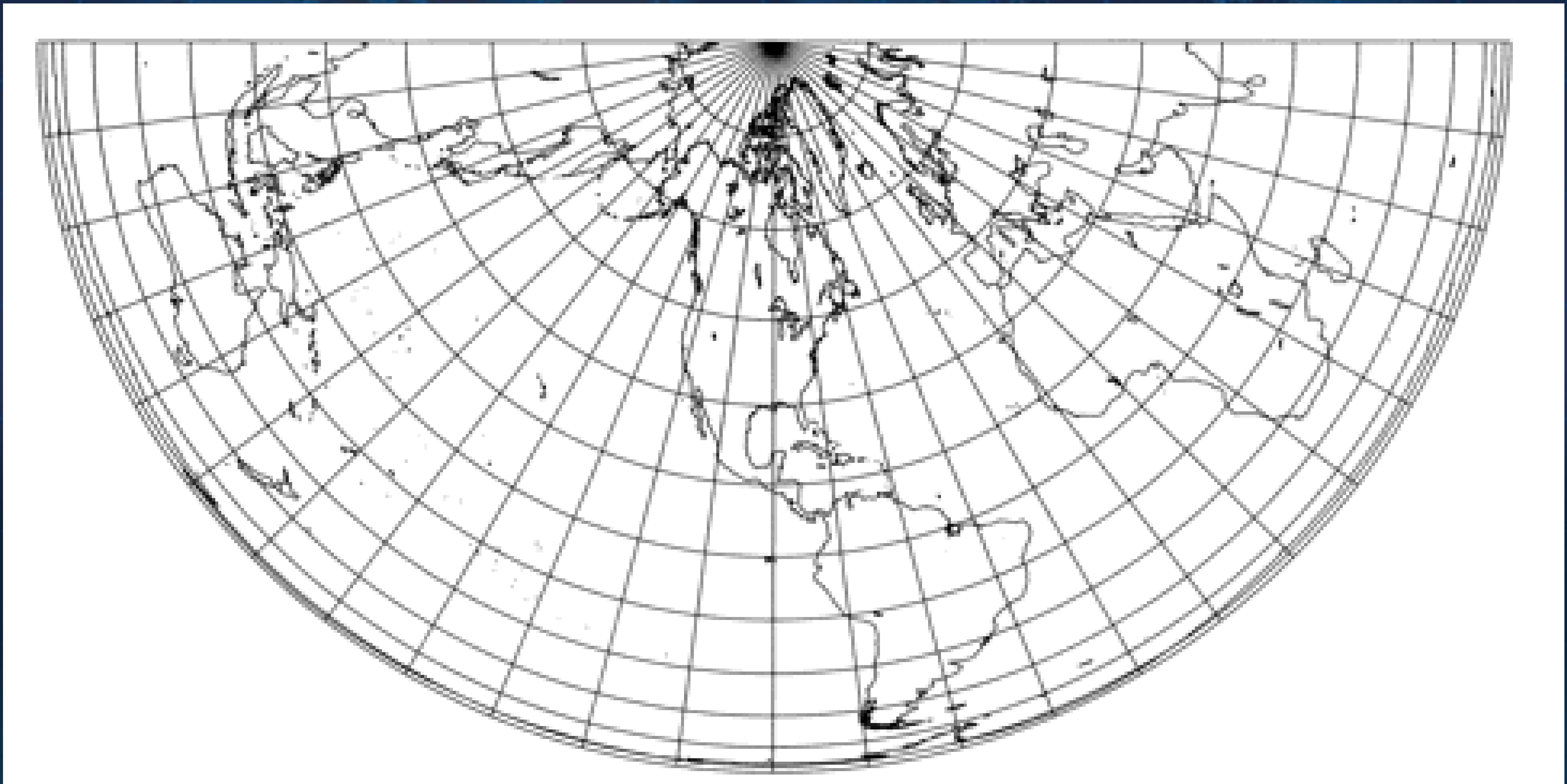
Equidistant conic projection (in meridians) with two undistorted parallels (de l'Isle)



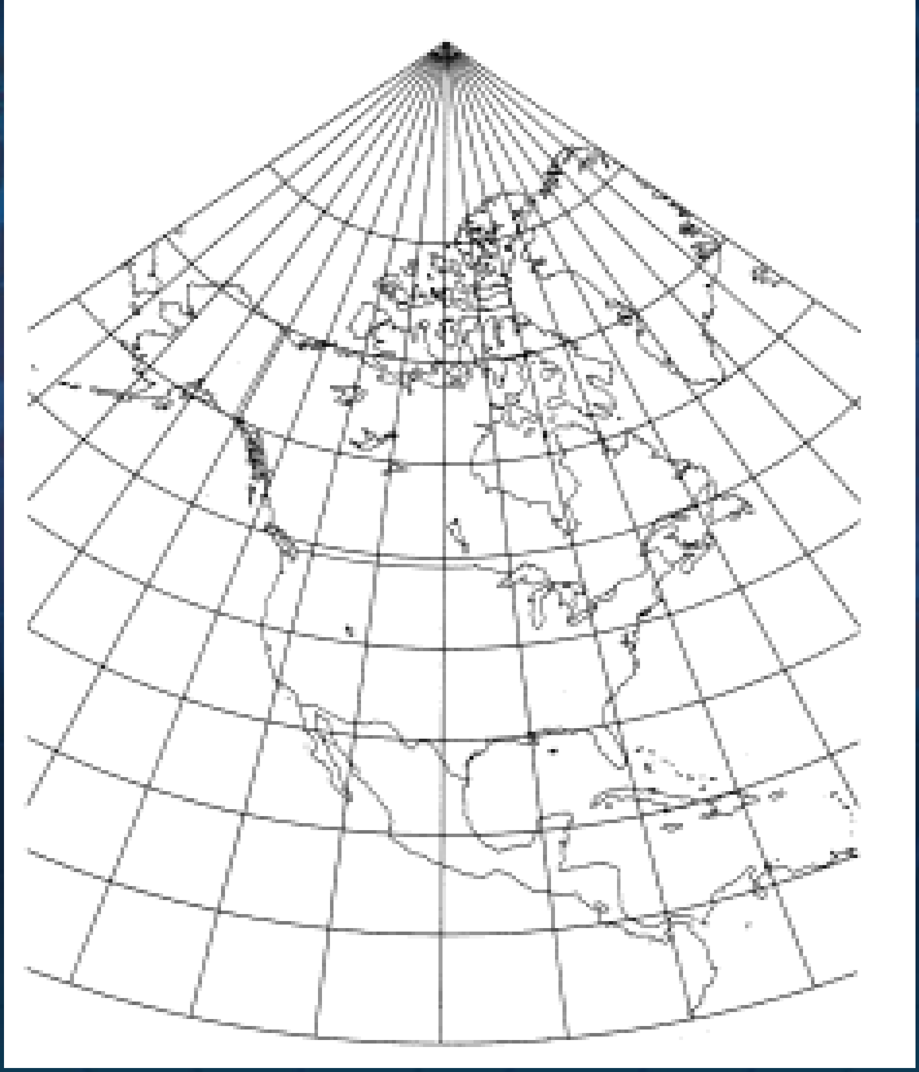
Equivalent conic projection (Albers) – 2 undistorted parallels



Equivalent conic (Lambert) projection – pole as point

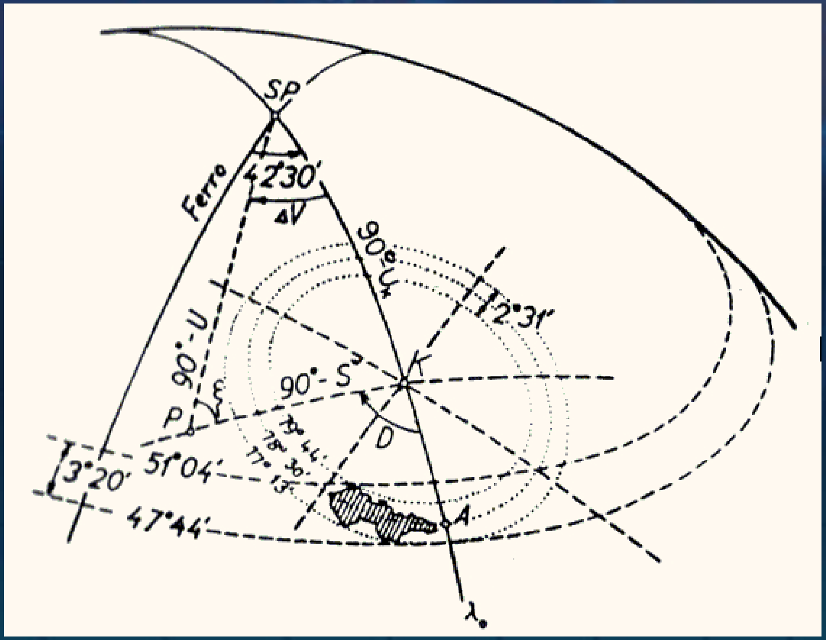


Conformal (Lambert) conic projection



Křovák's projection

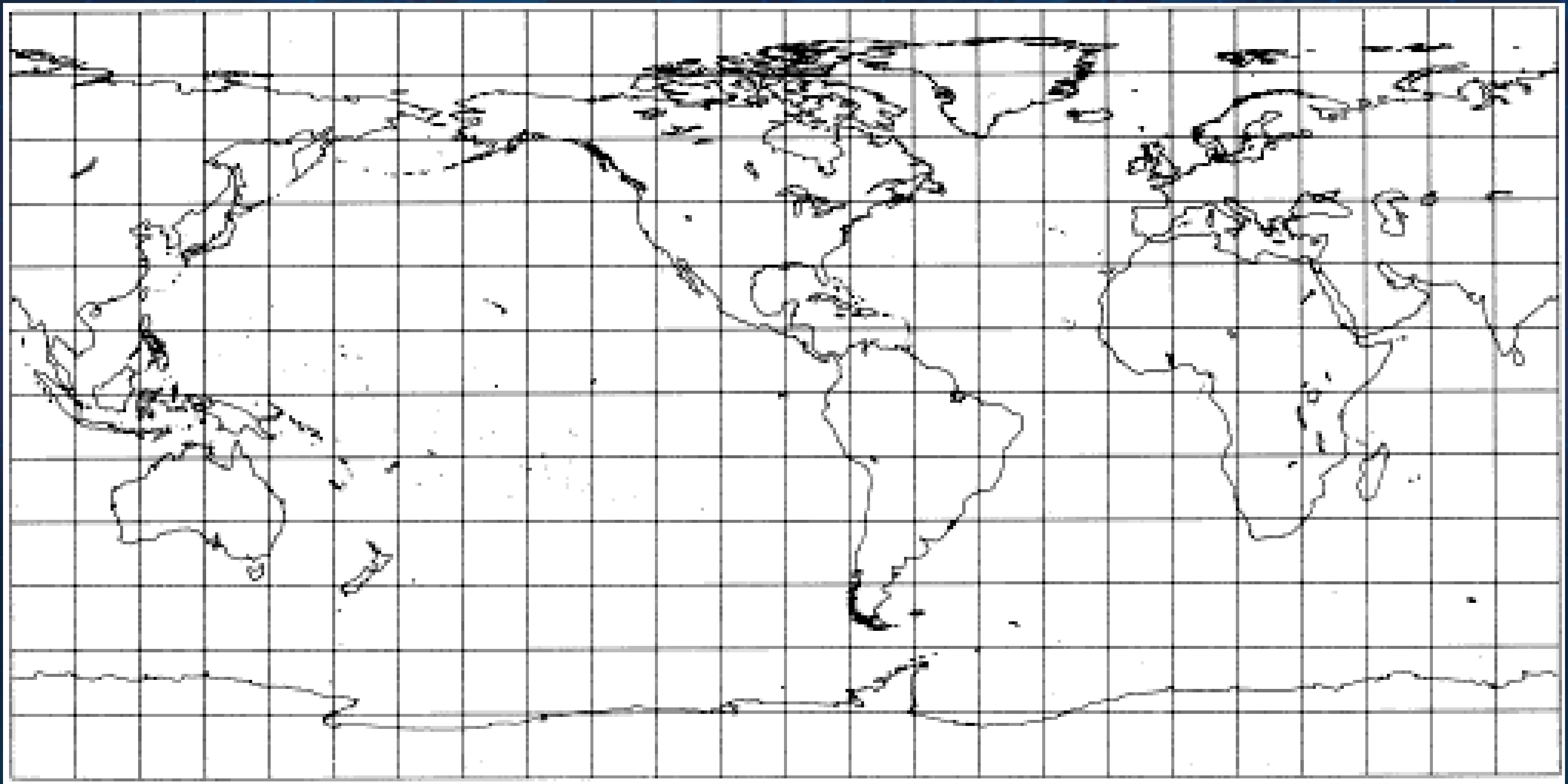
- double conformal conic projection in an oblique position
- author Ing. Josef Křovák (1922)
- became the basis of S–JTSK (a Czech system of a unified cadastral trigonometric network)



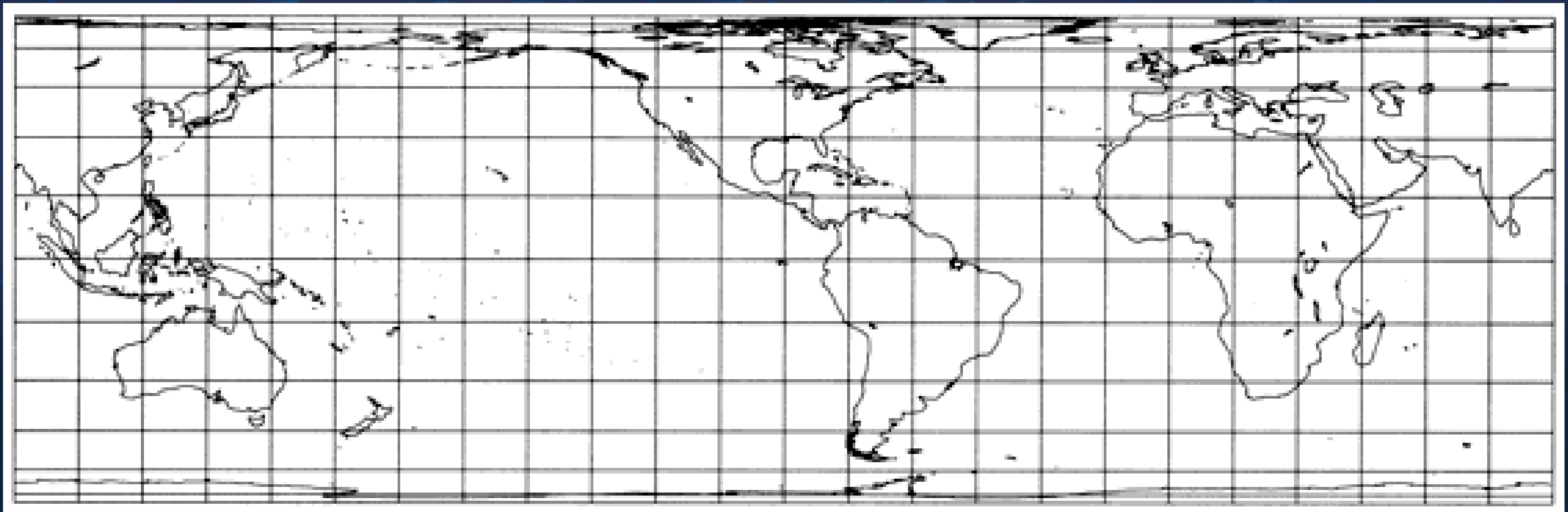
Cylindrical projections

- equator and parallels as parallel lines
- meridians parallel lines perpendicular to the parallels
- they are suitable in the transverse position for displaying longitudinal zones or in the normal position for the belt around the equator
- the smallest distortions are achieved around the tangent circle
- on the contrary, they are completely unsuitable for displaying polar regions
- with a suitable choice of the constant, we get individual cylindrical representations (equidistant, equivalent, conformal × number of undistorted parallels)

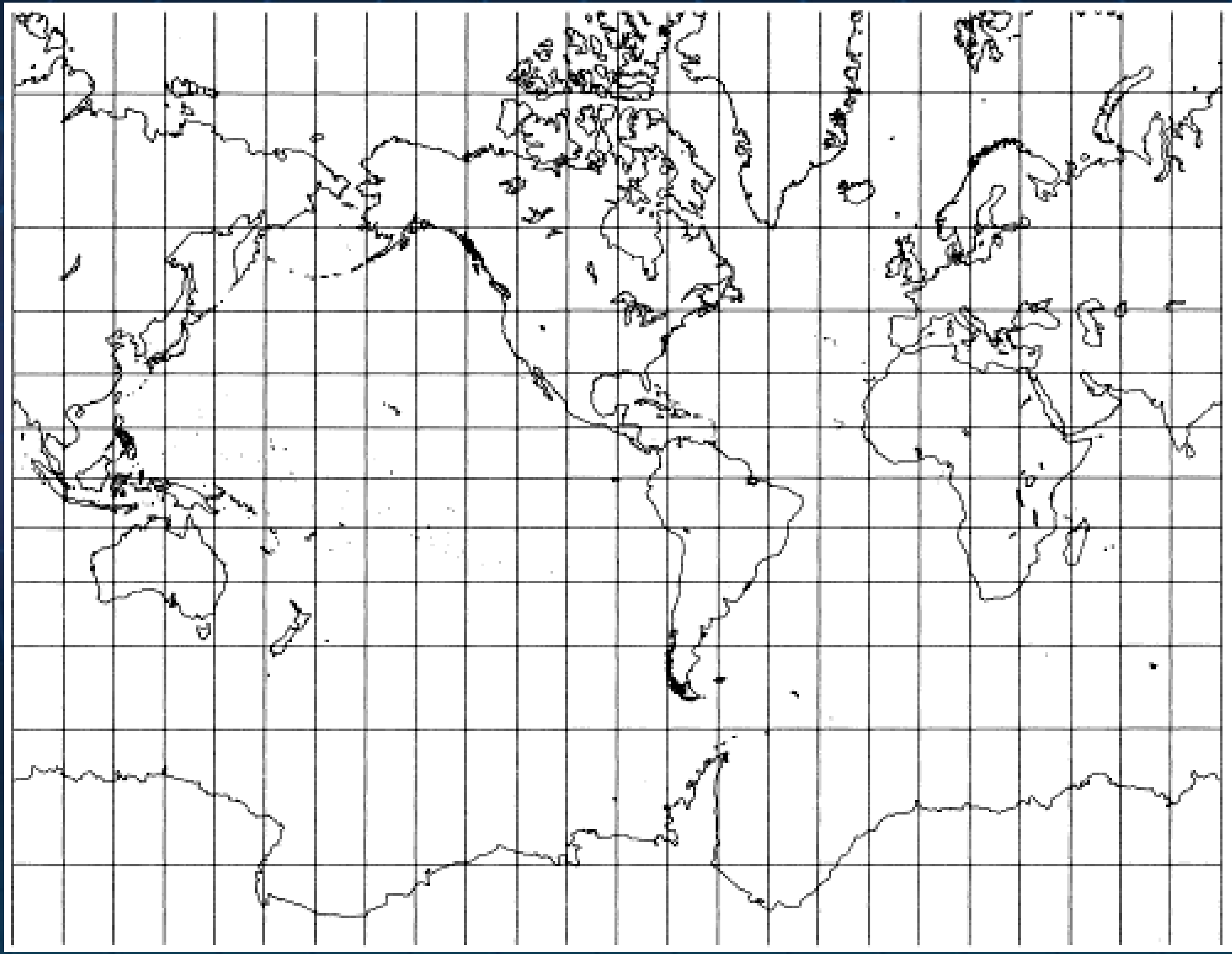
Equidistant cylindrical projection (in meridians) with one undistorted parallel (Marinus)



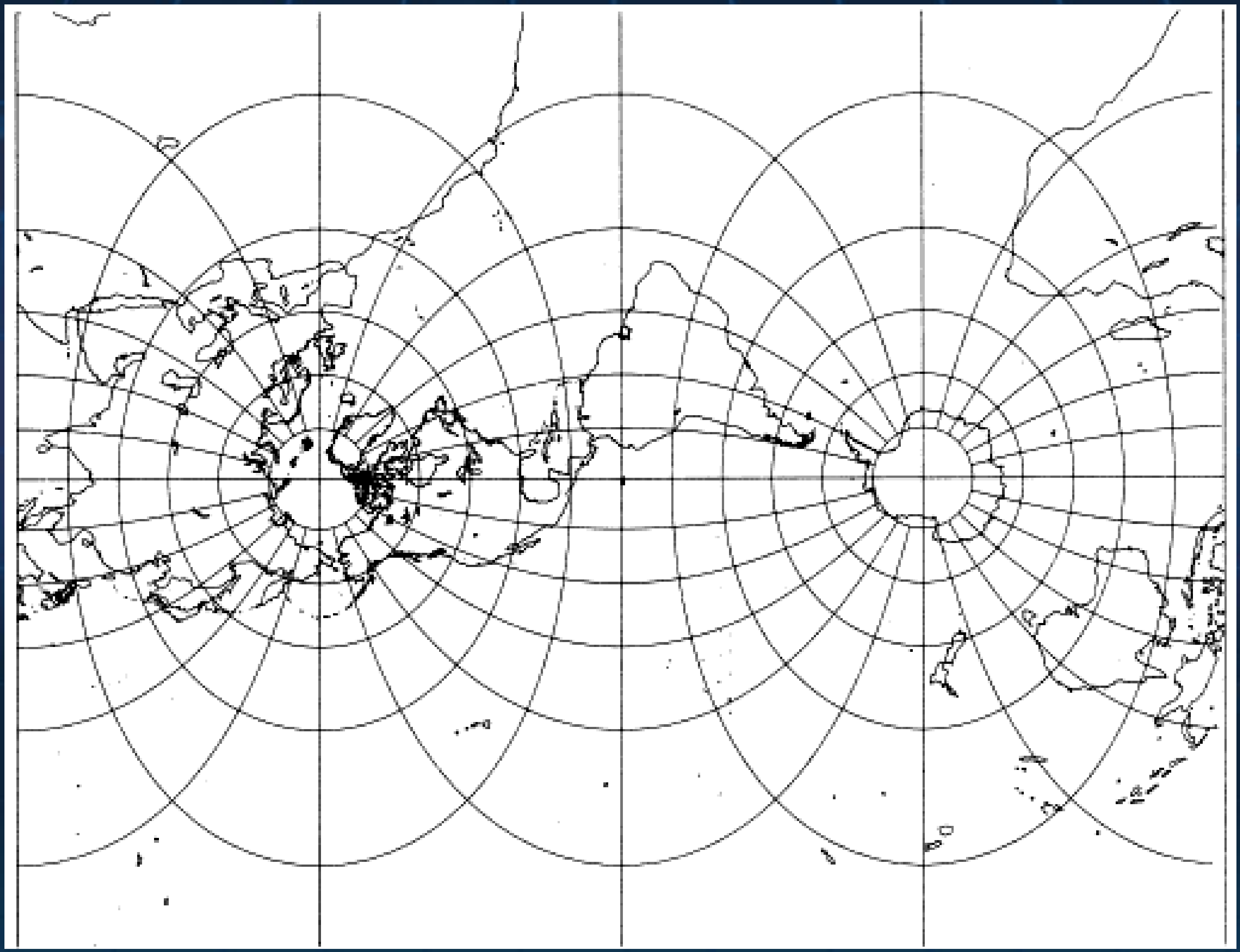
Equivalent cylindrical projection (Lambert)



Conformal cylindrical projection (Mercator)

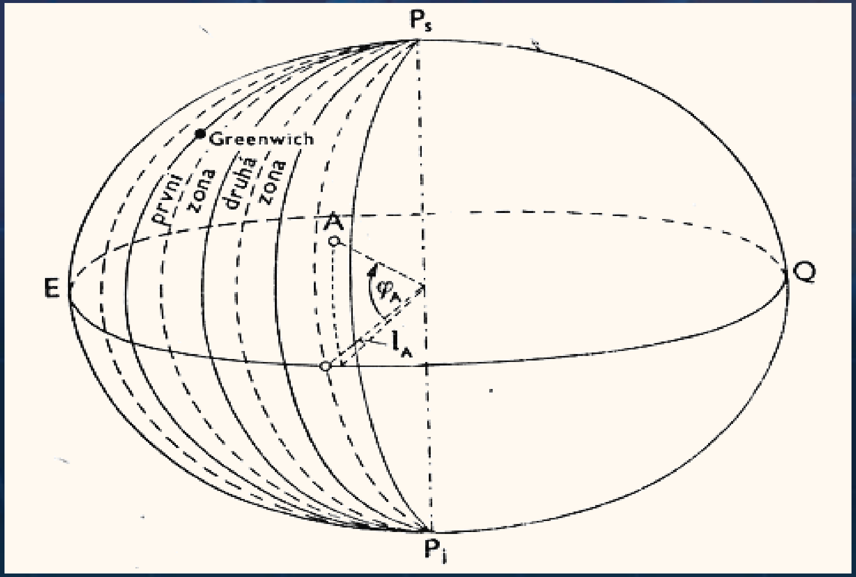


Conformal transverse cylindrical projection (Transverse Mercator)



Significant cylindrical projections

- **Conformal cylindrical projection (Gaussian)**
- **Transverse conformal cylindrical projection (UTM)**



Azimuthal projection

Geographic grid shape

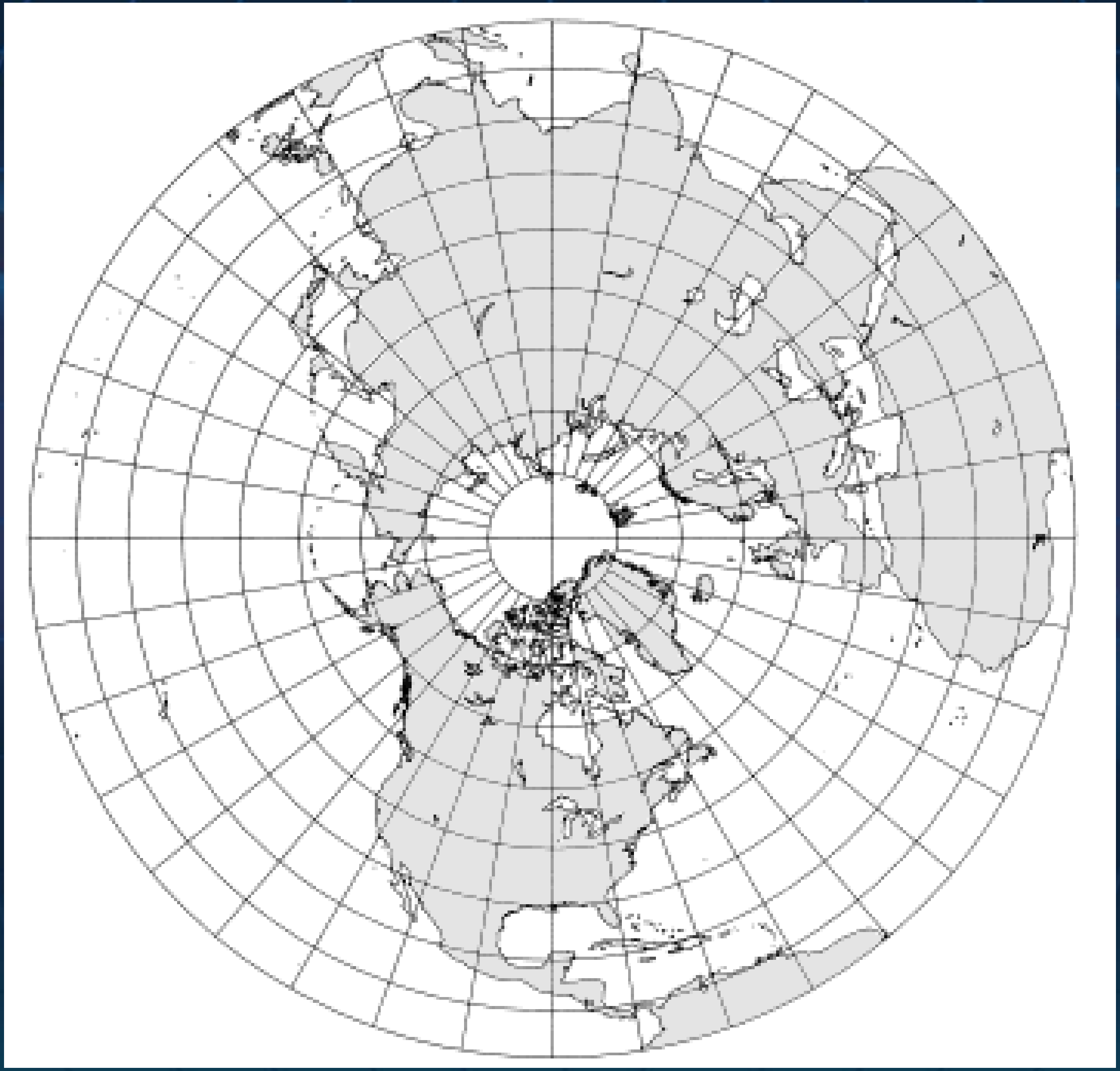
- meridians – a bundle of straight lines
- parallels – concentric circles
- pole – a circle or point

They are used for the territory around the (cartographic) pole, which is the center of the projection.

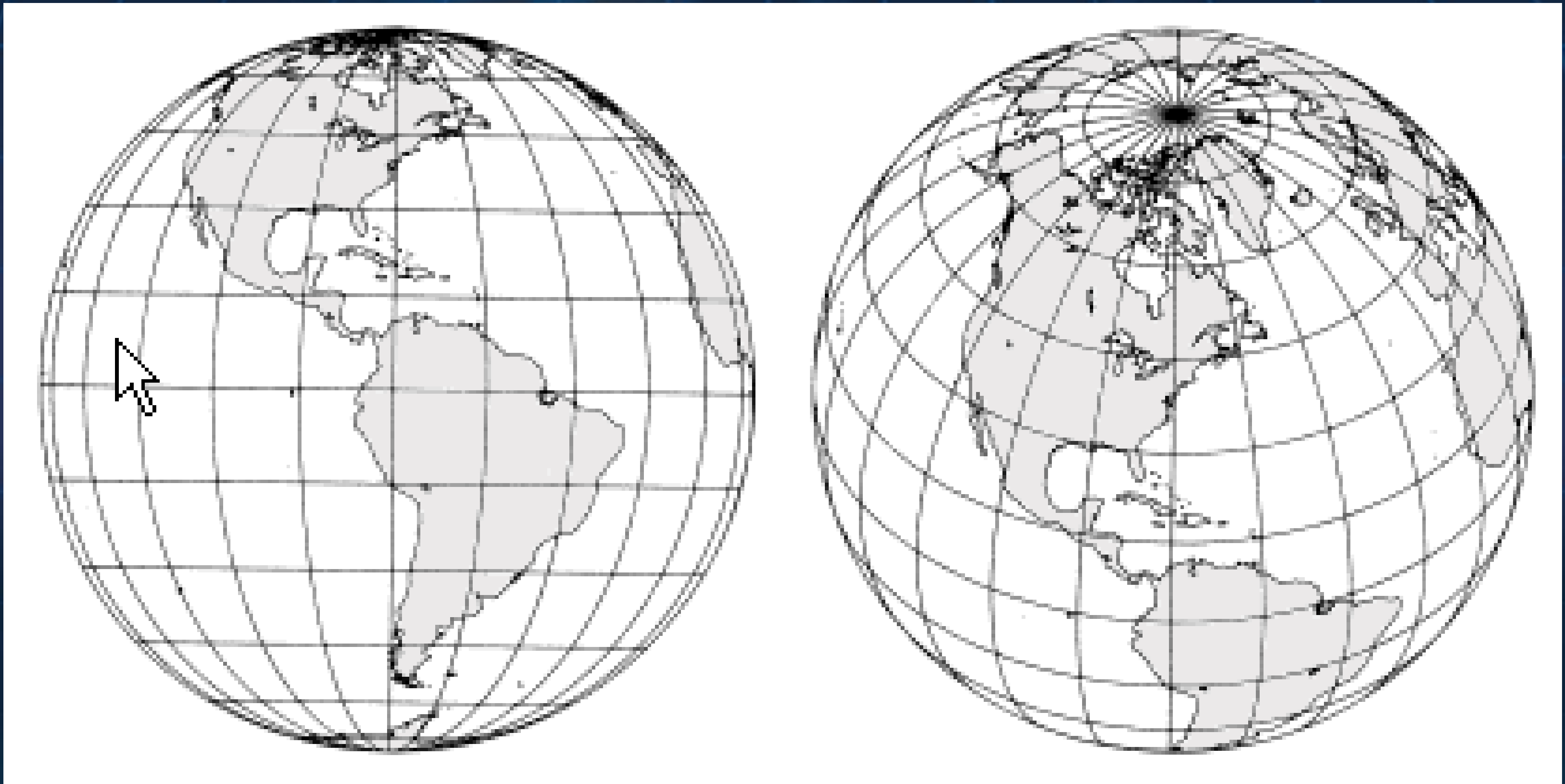
A number of them can be derived geometrically

With a suitable choice of the constant, we get individual azimuthal projections
(equidistant, equivalent, conformal
× pole rendition)

Equivalent azimuthal projection



Azimuthal orthographic projection



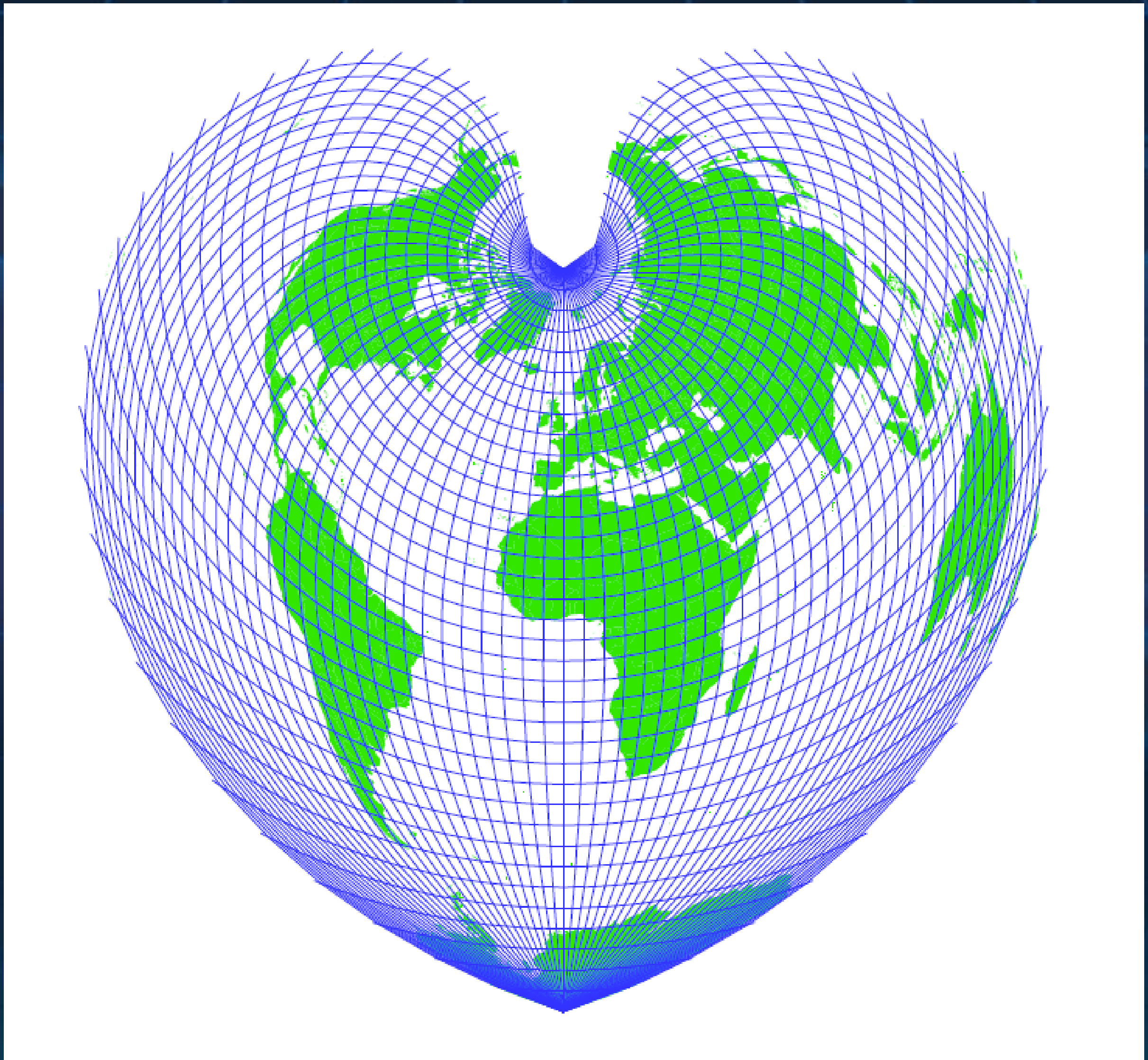
Pseudo-projections

- conic,
- cylindrical,
- azimuthal

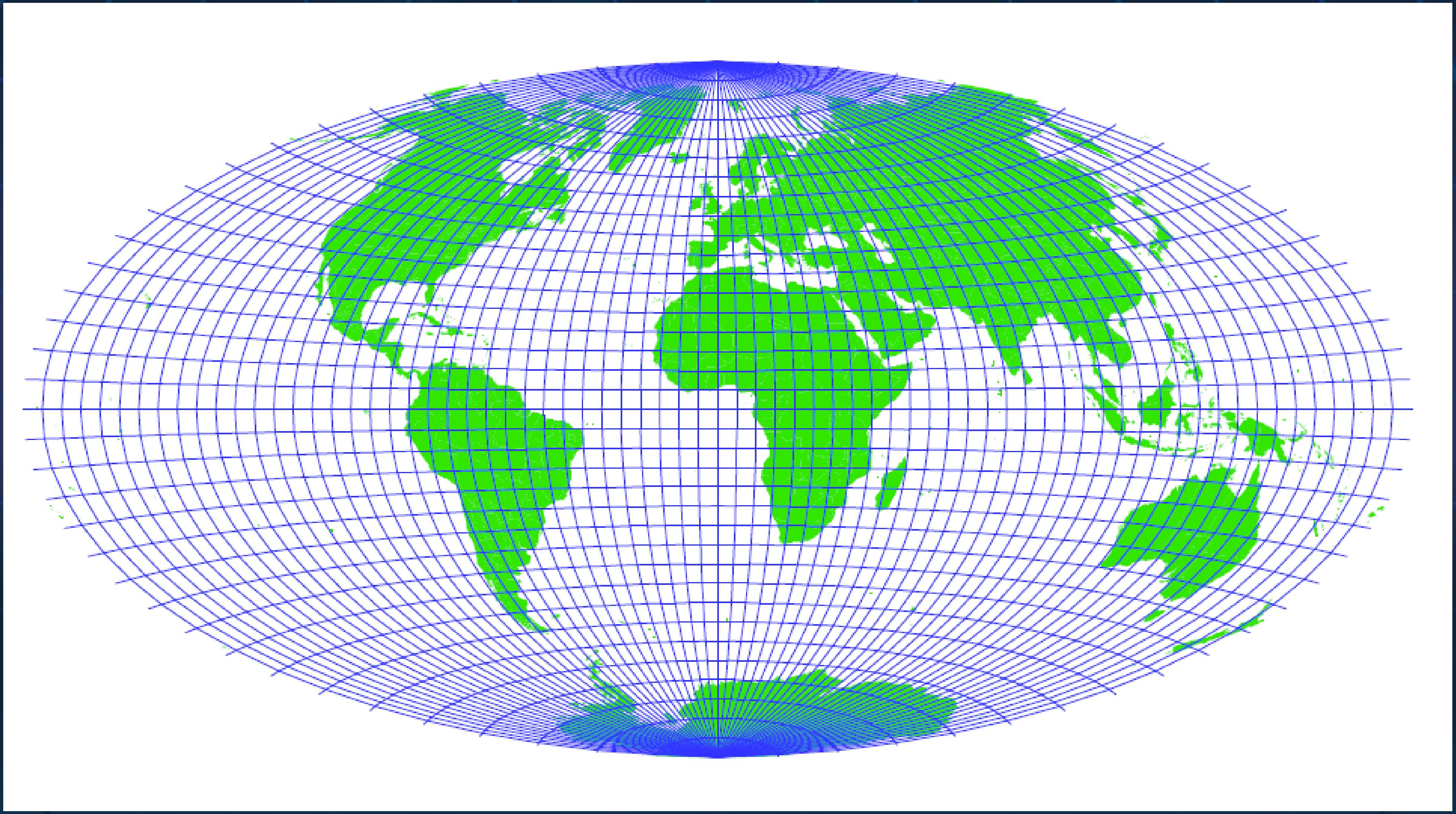
They are used for the world maps.

They are never conformal, but can be both equidistant and equivalent.

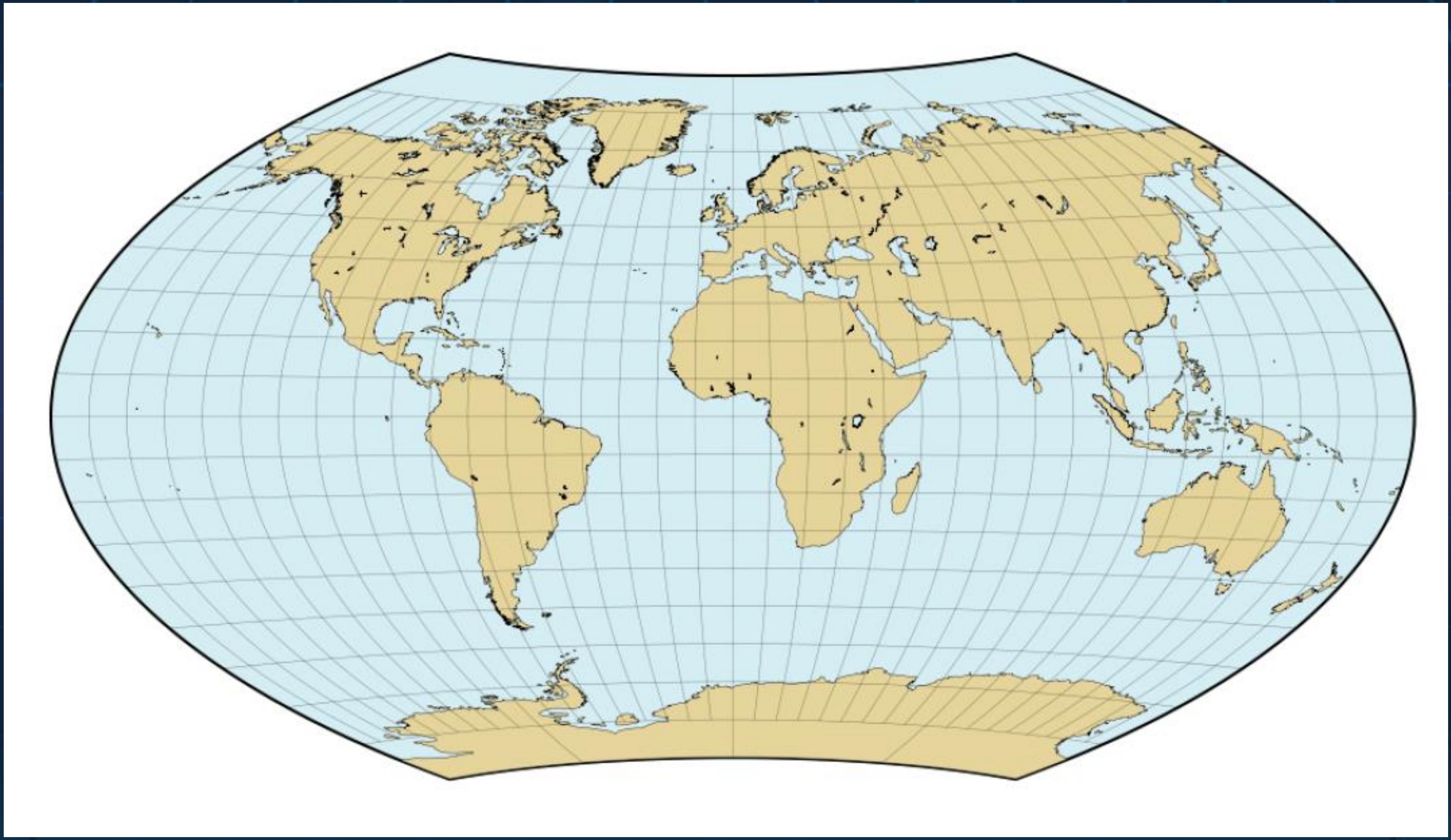
Bonne projection



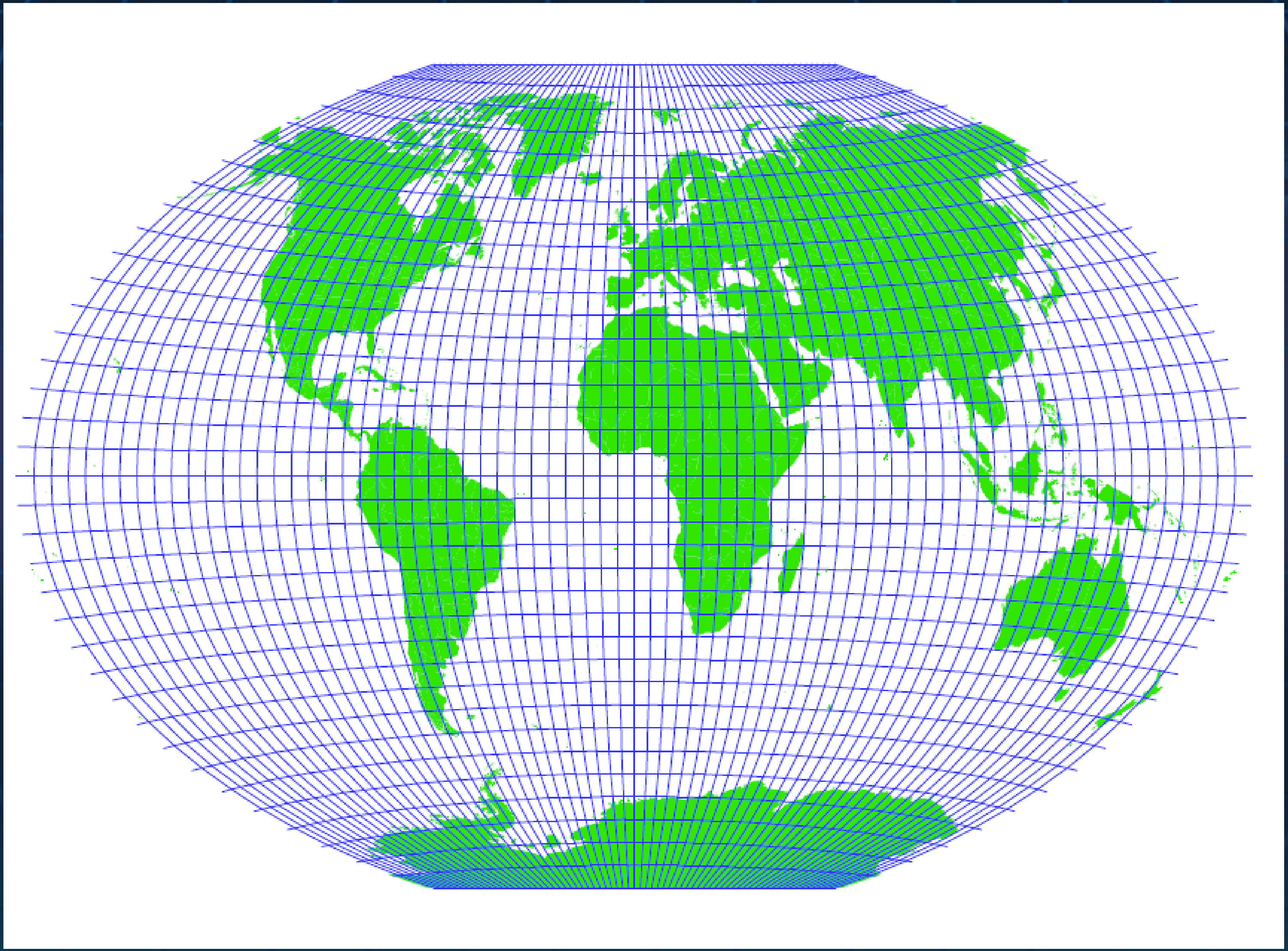
Aitoff projection



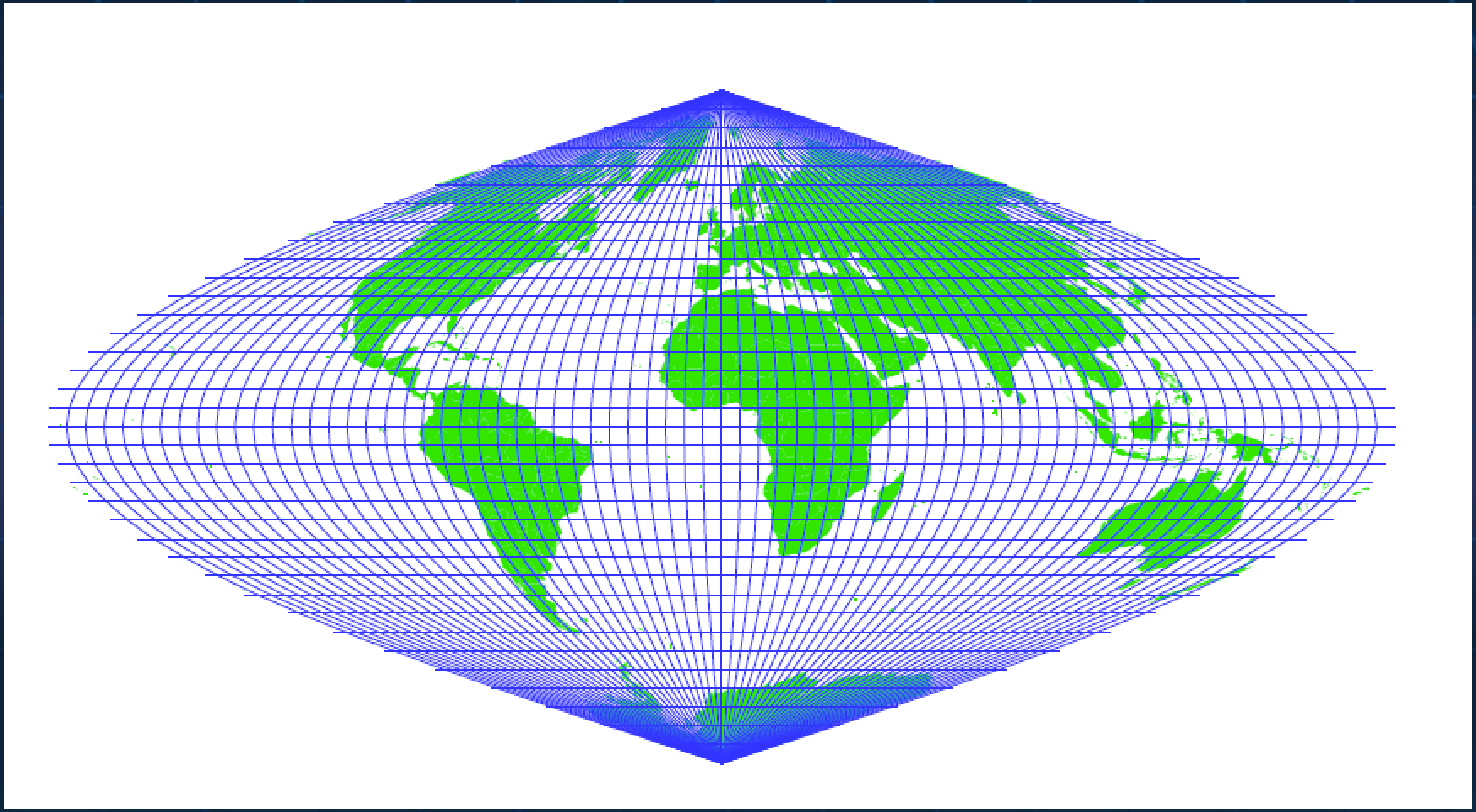
Wagner projection



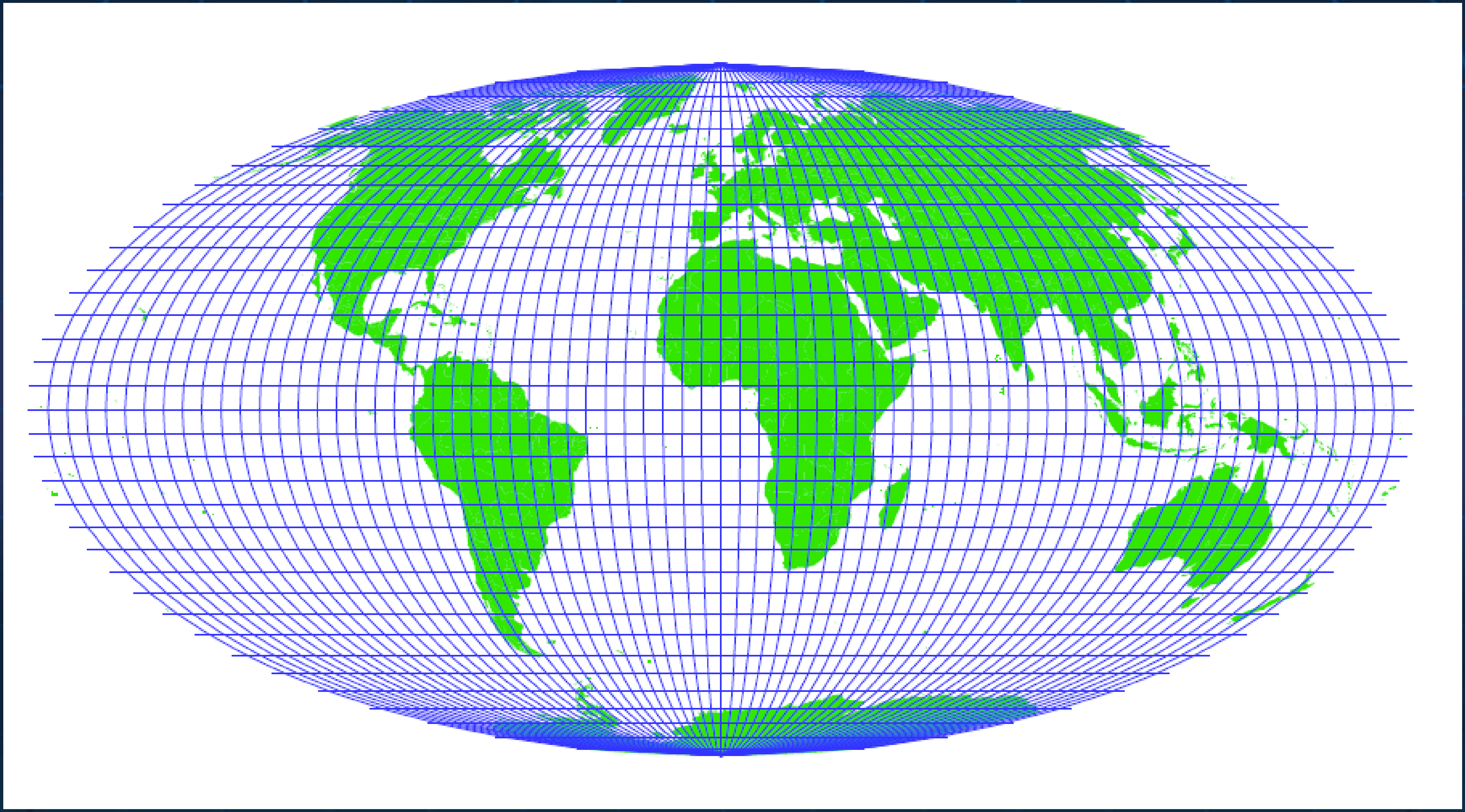
Winkel Tripel projection



Sinusoidal projection



Mollweid projection



Projections in GIS

- **Practical work**
  - EPSG database
  - Transformations
  - Work with geometry
- **Settings in ArcGIS**